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Vacuum string field theory ancestors of the GMS solitons

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Abstract: We define a sequence of VSFT D-branes whose low energy limit leads exactly to a corresponding sequence of GMS solitons. The D-branes are defined by acting on a fixed VSFT lump with operators defined by means of Laguerre polynomials whose argument is quadratic in the string creation operators. The states obtained in this way form an algebra under the SFT star product, which is isomorphic to a corresponding algebra of GMS solitons under the Moyal product. In order to obtain a regularized field theory limit we embed the theory in a constant background B field.

Keywords: Tachyon Condensation, String Field Theory, Non-Commutative Geometry.

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1. Introduction

The resemblance between Witten's star product [1] in open string field theory (SFT) and the Moyal product in noncommutative field theory is intriguing. There have been attempts to relate them to each other. It has been shown first of all that they are compatible: from the analysis of [2] and [3] it is clear that, if we restrict ourselves to the field theory regime in the presence of a constant background B field, the SFT $*$ product factorizes into the ordinary (Witten's) $*$ product and the Moyal product. On a different ground in [4, 5] it was suggested that SFT can be formulated in terms of Moyal products, although in an (unphysical) auxiliary space, see [6]–[9].

In this paper we would like to show that the similarity between the two products is deeper that one may suspect. In the course of the paper we will single out an interesting algebraic structure for a family of solitonic solutions of the Vacuum String Field Theory (VSFT), [10, 11]. This structure turns out to be exactly isomorphic to a corresponding one for a family of solitonic solutions in noncommutative field theory found by Gopakumar, Minwalla and Strominger, [12, 13].

The classical solutions of Vacuum String Field Theory (VSFT) factorize into a ghost part and a matter part. The ghost factor is universal, while the matter may take different forms, but they all satisfy the projector equation:

$$\Psi_m * \Psi_m = \Psi_m \,. \tag{1.1}$$

Well-known solutions are the *squeezed or sliver state solution* [14], which is interpreted as the D25-brane [11, 15], and the analogous lower dimensional tachyonic lumps. Other solutions have been found in [16, 17]. It has been shown in [22] that, in the field theory limit, the 23 dimensional lump solution can be identified with the simplest GMS soliton. However all the solutions that have been found so far reproduce, in the field theory limit, only the first and second simplest cases of the infinite series of (noncommutative) GMS solitons, [12, 13]. Since the GMS solitons can be thought of as solutions of the field theory limit of SFT, it is natural to imagine that there must exist a full series of SFT solutions which has not been found so far.

In this paper, starting from the squeezed state, we construct an infinite sequence of solutions to eq. (1.1), denoted $|\Lambda_n\rangle$ for any natural number n. $|\Lambda_n\rangle$ is generated by acting on a tachyonic lump solution $|\Lambda_0\rangle$ with $(-\kappa)^n L_n(\mathbf{x}/\kappa)$, where L_n is the n-th Laguerre polynomial, $\mathbf{x}$ is a quadratic expression in the string creation operators, see below eqs. (3.2), (3.3), and κ is an arbitrary constant. These states satisfy the remarkable properties

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{1.2}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle\,. \tag{1.3}$$

Each $|\Lambda_n\rangle$ represents a D23-brane, parallel to all the others. The field theory limit of $|\Lambda_n\rangle$ factors into the sliver state (D25-brane) and the n-th GMS soliton. The algebra (1.2) and the property (1.3) exactly reflect isomorphic properties of the GMS solitons (in terms of Moyal product). In other words, the GMS solitons are nothing but the relics of the $|\Lambda_n\rangle$ D23-branes in the low energy limit.

In the following, to avoid the singularities of the field theory limit of VSFT, [18], we introduce a constant background B field[1]. However it should be stressed that the proof of (1.2), (1.3) hold for any value of B.

The paper is organized as follows. In the next section we collect from the literature a series of results which are needed in the following. In section 3 we define the above solutions and prove eqs. (1.2), (1.3). In section 4 we find the field theory limit and recover the GMS solitons. Section 5 is devoted to a discussion of the above mentioned isomorphism. Finally, Appnedix A contains a discussion of the existence of our solutions, while appendix B is devoted to the $\alpha' \to 0$ limit.

2. A collection of formulas and results

To find in a natural way the field theory limit we introduce a background B field. This problem was studied in [19, 20, 21, 22]. In the last two references we found solutions to eq. (1.1) when a constant B field is turned on along some space directions. It will be enough to consider the simplest B field configuration, i.e. a B field nonvanishing only in

[1] One could implement the construction of this paper also without a B field. However, in the latter case, in the field theory limit one would have to introduce a regulator by hand, since, as shown in [18], this limit is singular. With a nonvanishing background B field we can avoid such an ad hoc procedure, as B itself provides a natural regularization.

two space directions, say the 24-th and 25-th ones. Let us denote these directions with the Lorentz indices α and β. Then, as is well-known [23], in these two direction we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta^{\alpha\beta}$. If we set

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{2.1}$$

with $B \geq 0$, they take the explicit form

$$G_{\alpha\beta} = \sqrt{\mathrm{DetG}}\,\delta_{\alpha\beta}\,, \qquad \theta^{\alpha\beta} = -(2\pi\alpha')^2 B\epsilon^{\alpha\beta}\,, \qquad \mathrm{DetG} = \left(1 + (2\pi\alpha' B)^2\right)^2, \tag{2.2}$$

where $\epsilon^{\alpha\beta}$ is the 2×2 antisymmetric symbol with $\epsilon^1{}_2 = 1$. α, β indices are raised and lowered using G.

The presence of the B field modifies the three-string vertex only in the 24-th and 25-th direction, which, in view of the D-brane interpretation, we call the transverse ones. After turning on the B-field the three-string vertex becomes

$$|V_3\rangle' = |V_{3,\perp}\rangle' \otimes |V_{3,||}\rangle \tag{2.3}$$

$|V_{3,||}\rangle$ is the same as in the ordinary case (without B field), while

$$|V_{3,\perp}\rangle' = K_2\, e^{-E'}|\tilde{0}\rangle_{123} \tag{2.4}$$

with

$$K_2 = \frac{\sqrt{2\pi b^3}}{A^2(4a^2+3)}(\mathrm{Det}G)^{1/4}\,, \tag{2.5}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN}a_N^{(s)\beta\dagger}\,. \tag{2.6}$$

We have introduced the indices $M = \{0, m\}, N = \{0, n\}$ and the vacuum $|\tilde{0}\rangle = |0\rangle \otimes |\Omega_{b,\theta}\rangle$, where $|\Omega_{b,\theta}\rangle$ is the vacuum with respect to the oscillators

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}\,, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}\,, \tag{2.7}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operator of the r-th string; i.e. $a_0^\alpha|\Omega_{b,\theta}\rangle = 0$. It is understood that $p^{(r)\alpha} = G^{\alpha\beta}p_\beta^{(r)}$, and

$$\left[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger}\right] = G^{\alpha\beta}\delta_{MN}\delta^{rs}\,. \tag{2.8}$$

The coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ are given by

$$\mathcal{V}^{\alpha\beta,rs}_{00} = G^{\alpha\beta}\delta^{rs} - \frac{2A^{-1}b}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\epsilon^{\alpha\beta}\chi^{rs}\right) \tag{2.9}$$

$$\mathcal{V}^{\alpha\beta,rs}_{0n} = \frac{2A^{-1}\sqrt{b}}{4a^2+3}\sum_{t=1}^{3}\left(G^{\alpha\beta}\phi^{rt} - ia\epsilon^{\alpha\beta}\chi^{rt}\right)V^{ts}_{0n} \tag{2.10}$$

$$\mathcal{V}^{\alpha\beta,rs}_{mn} = G^{\alpha\beta}V^{rs}_{mn} - \frac{2A^{-1}}{4a^2+3}\sum_{t,v=1}^{3}V^{rv}_{m0}\left(G^{\alpha\beta}\phi^{vt} - ia\epsilon^{\alpha\beta}\chi^{vt}\right)V^{ts}_{0n}\,. \tag{2.11}$$

Here, by definition, $V_{0n}^{rs} = V_{n0}^{sr}$, and

$$\phi^{rs} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}, \qquad \chi^{rs} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}. \tag{2.12}$$

These two matrices satisfy the algebra

$$\chi^2 = -2\phi\,, \qquad \phi\chi = \chi\phi = \frac{3}{2}\chi\,, \qquad \phi^2 = \frac{3}{2}\phi\,. \tag{2.13}$$

Moreover, in (2.11), we have introduced the notation

$$A = V_{00} + \frac{b}{2}\,, \qquad a = -\frac{\pi^2}{A}\,B\,. \tag{2.14}$$

Next we introduce the twist matrix C' by $C'_{MN} = (-1)^M \delta_{MN}$ and define

$$\mathcal{X}^{rs} \equiv C'\mathcal{V}^{rs}\,, \qquad r,s = 1,2\,, \qquad \mathcal{X}^{11} \equiv \mathcal{X}\,. \tag{2.15}$$

These matrices commute

$$[\mathcal{X}^{rs}, \mathcal{X}^{r's'}] = 0\,. \tag{2.16}$$

Moreover we have the following properties, which mark a difference with the $B = 0$ case,

$$C'\mathcal{V}^{rs} = \tilde{\mathcal{V}}^{sr}C'\,, \qquad C'\mathcal{X}^{rs} = \tilde{\mathcal{X}}^{sr}C' \tag{2.17}$$

where tilde denotes transposition with respect to the α, β indices alone. Finally one can prove that

$$\begin{aligned} \mathcal{X} + \mathcal{X}^{12} + \mathcal{X}^{21} &= \mathbb{I} \\ \mathcal{X}^{12}\mathcal{X}^{21} &= \mathcal{X}^2 - \mathcal{X} \\ (\mathcal{X}^{12})^2 + (\mathcal{X}^{21})^2 &= \mathbb{I} - \mathcal{X}^2 \\ (\mathcal{X}^{12})^3 + (\mathcal{X}^{21})^3 &= 2\mathcal{X}^3 - 3\mathcal{X}^2 + \mathbb{I}\,. \end{aligned} \tag{2.18}$$

In the matrix products of these identities, as well as throughout the paper, the indices α, β must be understood in alternating up/down position: $\mathcal{X}^\alpha{}_\beta$. For instance, in (2.18) $\mathbb{I}$ stands for $\delta^\alpha{}_\beta\,\delta_{MN}$.

The lump solution we found in [21] satisfies $|\mathcal{S}\rangle = |\mathcal{S}\rangle * |\mathcal{S}\rangle$ and can be written as

$$\begin{aligned} |\mathcal{S}\rangle = &\left\{\mathrm{Det}(1 - X)^{1/2}\mathrm{Det}(1 + T)^{1/2}\right\}^{24} \exp\left(-\frac{1}{2}\eta_{\bar{\mu}\bar{\nu}} \sum_{m,n\geq 1} a_m^{\bar{\mu}\dagger} S_{mn} a_n^{\bar{\nu}\dagger}\right)|0\rangle \otimes \\ &\otimes \frac{A^2(3 + 4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}} \left(\mathrm{Det}(\mathbb{I} - \mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I} + \mathcal{T})^{1/2}\right) \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle\,. \end{aligned} \tag{2.19}$$

The quantities in the first line are defined in ref. [11] with $\bar{\mu}, \bar{\nu} = 0, \ldots, 23$ denoting the parallel directions to the lump. The matrix $\mathcal{S} = C'\mathcal{T}$ is given by

$$\mathcal{T} = \frac{1}{2\mathcal{X}} \left(\mathbb{I} + \mathcal{X} - \sqrt{(\mathbb{I} + 3\mathcal{X})(\mathbb{I} - \mathcal{X})} \right). \tag{2.20}$$

This is a solution to the equation

$$\mathcal{X}\mathcal{T}^2 - (\mathbb{I} + \mathcal{X})\mathcal{T} + \mathcal{X} = 0\,. \tag{2.21}$$

Another ingredient we need in order to construct new solutions starting from (2.19) are projectors similar to those introduced in [16]. They are defined only along the transverse directions by

$$\rho_1 = \frac{1}{(\mathbb{I} + \mathcal{T})(\mathbb{I} - \mathcal{X})} \left[\mathcal{X}^{12}(\mathbb{I} - \mathcal{T}\mathcal{X}) + \mathcal{T}\left(\mathcal{X}^{21}\right)^2 \right] \tag{2.22}$$

$$\rho_2 = \frac{1}{(\mathbb{I} + \mathcal{T})(\mathbb{I} - \mathcal{X})} \left[\mathcal{X}^{21}(\mathbb{I} - \mathcal{T}\mathcal{X}) + \mathcal{T}\left(\mathcal{X}^{12}\right)^2 \right]. \tag{2.23}$$

They satisfy

$$\rho_1^2 = \rho_1\,, \qquad \rho_2^2 = \rho_2\,, \qquad \rho_1 + \rho_2 = \mathbb{I} \tag{2.24}$$

i.e. they project onto orthogonal subspaces. Moreover, if we use the superscript ^{T} to denote transposition with respect to the indices N, M and α, β, we have

$$\rho_1^T = \tilde{\rho}_1 = C'\rho_2 C'\,, \qquad \rho_2^T = \tilde{\rho}_2 = C'\rho_1 C'\,. \tag{2.25}$$

With all these ingredients we can now move on, give a precise definition of the $|\Lambda_n\rangle$ states and demonstrate the properties announced in the introduction.

3. The states $|\Lambda_n\rangle$ and their properties

To define the states $|\Lambda_n\rangle$ we start from the lump solution (2.19). i.e. we take $|\Lambda_0\rangle = |\mathcal{S}\rangle$. However, in the following, we will limit ourselves only to the transverse part of it, the parallel one being universal and irrelevant for our construction. We will denote the transverse part by $|\mathcal{S}_\perp\rangle$.

First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_1\xi = 0\,, \qquad \rho_2\xi = \xi\,, \qquad \text{and} \quad \rho_1\zeta = 0\,, \qquad \rho_2\zeta = \zeta\,. \tag{3.1}$$

Next we define

$$\mathbf{x} = \left(a^\dagger \tau \xi\right)\left(a^\dagger C' \zeta\right) = \left(a_N^{\alpha\dagger} \tau_\alpha{}^\beta \xi_{N\beta}\right)\left(a_N^{\alpha\dagger} C'_{NM} \zeta_{M\alpha}\right) \tag{3.2}$$

where τ is the matrix $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and introduce the Laguerre polynomials

$L_n(\mathbf{x}/\kappa)$. The definition of $|\Lambda_n\rangle$ is as follows

$$|\Lambda_n\rangle = (-\kappa)^n L_n\left(\frac{\mathbf{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle \tag{3.3}$$

where κ is an arbitrary constant. Hermiticity requires that

$$(a\tau\xi^*)(aC'\zeta^*) = (a\tau C'\xi)(a\zeta) \tag{3.4}$$

and that κ be a real constant. The superscript * denotes complex conjugation.

Finally we impose that the two following conditions be satisfied

$$\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = -1\,, \qquad \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = -\kappa\,. \tag{3.5}$$

Compatibility of eqs. (3.1), (3.4), (3.5) is discussed in appendix A. It is shown there that a solution to (3.4) compatible with the other equations is

$$\zeta = \tau\xi^*\,. \tag{3.6}$$

Before we set out to prove the properties of these states $|\Lambda_n\rangle$, let us spend a few words to motivate their definition. The definition (3.3) is not, as one might suspect, dictated in the first place by the similarity with the form of the GMS solitons. Rather it has been selected due to its apparently unique role in the framework of Witten's star algebra.

In [22], on the wake of [16], starting from the (transverse) lump solution $|\mathcal{S}_\perp\rangle$ we introduced a new lump solution $|\mathcal{P}_\perp\rangle = (\mathbf{x}-\kappa)|\mathcal{S}_\perp\rangle$. Imposing that $|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle = |\mathcal{P}_\perp\rangle$ and $|\mathcal{P}_\perp\rangle * |\mathcal{S}_\perp\rangle = 0$ and, moreover, that $\langle\mathcal{P}_\perp|\mathcal{P}_\perp\rangle = \langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle$, we found the conditions (3.5).

The next most complicated state one is lead to try is of the form

$$|\mathcal{P}'\rangle = \left(\alpha + \beta\mathbf{x} + \gamma\mathbf{x}^2\right)|\mathcal{S}_\perp\rangle\,. \tag{3.7}$$

The conditions this state has to satisfy turn out to be more restrictive than for $|\mathcal{P}\rangle$, but, nevertheless, are satisfied if, besides conditions (3.5), the following relations hold

$$-2(\alpha)^{1/2} = \beta\,, \qquad \gamma = \frac{1}{2} \tag{3.8}$$

and then, putting $\alpha = \kappa$

$$|\mathcal{P}'\rangle = \left(\kappa^2 - 2\kappa\mathbf{x} + \frac{1}{2}\mathbf{x}^2\right)|\mathcal{S}_\perp\rangle\,. \tag{3.9}$$

The polynomial in the r.h.s. is nothing but the second Laguerre polynomial of $\mathbf{x}/\kappa$ multiplied by κ^2. We deduce from this that the Laguerre polynomials must play a fundamental role in this problem and, as a consequence, put forward the general ansatz (3.3).

Proving the necessity of the conditions (3.5) for general n is very cumbersome, so we will limit ourselves to showing that these conditions are sufficient. However it is instructive and rather easy to see, at least, that the second condition (3.5) is necessary in general. In fact, by requiring that the state $|\Lambda_n\rangle$ be orthogonal to the 'ground state' $|\mathcal{S}_\perp\rangle$, we get:

$$
\begin{aligned}
|\Lambda_n\rangle * |\mathcal{S}_\perp\rangle &= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} \frac{(-\mathbf{x}/\kappa)^j}{j!} |\mathcal{S}_\perp\rangle * |\mathcal{S}_\perp\rangle \\
&= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} (\kappa)^{-j} \times \\
&\quad \times (\xi\tau C')^{\alpha_1}_{l_1} \dots (\xi\tau C')^{\alpha_j}_{l_j} \zeta^{\beta_1}_{k_1} \dots \zeta^{\beta_j}_{k_j} \frac{\partial}{\partial\mu^{\alpha_1}_{l_1}} \cdots \frac{\partial}{\partial\mu^{\alpha_j}_{l_j}} \frac{\partial}{\partial\mu^{\beta_1}_{k_1}} \cdots \frac{\partial}{\partial\mu^{\beta_j}_{k_j}} \times \\
&\quad \times \exp\Big(- (\chi^T \mathcal{K}_1)^{-1}\mu - \frac{1}{2}\mu^T(\mathcal{V}\mathcal{K}^{-1})_{11}\mu\Big) |\mathcal{S}_\perp\rangle\Big|_{\mu=0} \\
&= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} (\kappa)^{-j} \left(\xi^T\tau \frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta\right)^j |\mathcal{S}\perp\rangle \\
&= (-\kappa)^n \left(1 + \frac{1}{\kappa}\xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta\right)^n |\mathcal{S}_\perp\rangle = 0
\end{aligned}
\tag{3.10}
$$

which is true for the choice κ given by the second eq. (3.5). In order to obtain this result we have followed ref. [16] (see also [22]).

3.1 Proof of eq. (1.2)

The star product $|\Lambda_n\rangle * |\Lambda_{n'}\rangle$ can be evaluated by using the explicit expression of the Laguerre polynomials

$$
|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \left((-\kappa)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-\mathbf{x}/\kappa)^k}{k!} |\mathcal{S}_\perp\rangle \right) * \left((-\kappa)^{n'} \sum_{p=0}^{n'} \binom{n'}{p} \frac{(-\mathbf{x}/\kappa)^p}{p!} |\mathcal{S}_\perp\rangle \right)
\tag{3.11}
$$

Therefore we need to compute $(\mathbf{x}^k|\mathcal{S}_\perp\rangle) * (\mathbf{x}^p|\mathcal{S}_\perp\rangle)$. According to [16], this is given by

$$
\begin{aligned}
\Big(\mathbf{x}^k|\mathcal{S}_\perp\rangle\Big) * (\mathbf{x}^p|\mathcal{S}_\perp\rangle) = (\xi\tau C')^{\alpha_1}_{l_1} &\dots (\xi\tau C')^{\alpha_k}_{l_k} \zeta^{\beta_1}_{j_1} \dots \zeta^{\beta_k}_{j_k} \frac{\partial}{\partial\mu^{\alpha_1}_{l_1}} \cdots \frac{\partial}{\partial\mu^{\alpha_k}_{l_k}} \frac{\partial}{\partial\mu^{\beta_1}_{j_1}} \cdots \frac{\partial}{\partial\mu^{\beta_k}_{j_k}} \times \\
&\times (\xi\tau C')^{\bar\alpha_1}_{\bar l_1} \dots (\xi\tau C')^{\bar\alpha_p}_{\bar l_p} \zeta^{\bar\beta_1}_{\bar j_1} \dots \zeta^{\bar\beta_p}_{\bar j_p} \frac{\partial}{\partial\bar\mu^{\bar\alpha_1}_{\bar l_1}} \cdots \frac{\partial}{\partial\bar\mu^{\bar\alpha_p}_{\bar l_p}} \frac{\partial}{\partial\bar\mu^{\bar\beta_1}_{\bar j_1}} \cdots \frac{\partial}{\partial\bar\mu^{\bar\beta_p}_{\bar j_p}} \times \\
&\times \exp\Big(- \chi^T\mathcal{K}^{-1}M - \frac{1}{2}M^T\mathcal{V}\mathcal{K}^{-1}M\Big) |\mathcal{S}_\perp\rangle)\Big|_{\mu=\bar\mu=0}
\end{aligned}
\tag{3.12}
$$

where

$$
\mathcal{K} = \mathbb{I} - \mathcal{T}\mathcal{X}\,, \qquad \mathcal{V} = \begin{pmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{pmatrix}
\tag{3.13}
$$

and

$$
M = \begin{pmatrix} \mu \\ \bar\mu \end{pmatrix}, \qquad \chi^T = \Big(a^\dagger\mathcal{V}^{12}, a^\dagger\mathcal{V}^{21}\Big)\,, \qquad \chi^T\mathcal{K}^{-1}M = a^\dagger C'(\rho_1\mu + \rho_2\bar\mu)\,.
\tag{3.14}
$$

The explicit computation, at first sight, looks daunting. However, we may avail ourselves of the following identities

$$
\begin{aligned}
\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{\alpha\alpha} \zeta &= \xi^T \tau C' \left(\mathcal{V}\mathcal{K}^{-1}\right)_{\alpha\alpha} \tau C' \zeta = \xi^T C' \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta = 0 \\
\xi^T \tau C' \left(\mathcal{V}\mathcal{K}^{-1}\right)_{\alpha\alpha} \zeta &= \xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{\alpha\alpha} \tau C' \zeta = \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta = -\kappa
\end{aligned}
\tag{3.15}
$$

for $\alpha = 1, 2$, and

$$
\begin{aligned}
\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \zeta &= \xi^T \tau C' \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \tau C' \zeta = -\xi^T C' \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta = 0 \\
\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \zeta &= \xi^T \tau C' \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \tau C' \zeta = \xi^T C' \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta = 0 \\
\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \tau C' \zeta &= \xi^T \tau C' \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \zeta = \xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta = -1 \\
\xi^T \tau C' \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \zeta &= \xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \tau C' \zeta = -\xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta = \kappa
\end{aligned}
\tag{3.16}
$$

Moreover

$$
\begin{aligned}
&\left(\chi^T \mathcal{K}^{-1}\right)_1 \xi = 0\,, \qquad \left(\chi^T \mathcal{K}^{-1}\right)_1 \tau C' \xi = a^\dagger \tau \xi \\
&\left(\chi^T \mathcal{K}^{-1}\right)_2 \xi = a^\dagger C' \xi\,, \qquad \left(\chi^T \mathcal{K}^{-1}\right)_2 \tau C' \xi = 0
\end{aligned}
\tag{3.17}
$$

with analogous equations for ζ.

In evaluating (3.15), (3.16), (3.17) we have used the methods of ref. [16] (see also [22]), together with eqs. (3.1), (3.5). These results are all we need to explicitly compute (3.12). In fact it is easy to verify that the latter can be mapped to a rather simple combinatorial problem. To show this we introduce generic variables $x, y, \bar{x}, \bar{y}$, and make the following formal replacements:

$$
\begin{aligned}
A &\equiv \chi^T \mathcal{K}^{-1} M \longrightarrow x\left(a^\dagger \tau \xi\right) + \bar{y}\left(a^\dagger C' \zeta\right), \\
B &\equiv M^T \mathcal{V}\mathcal{K}^{-1} M \longrightarrow \left(-\kappa x y + \kappa x \bar{y} - \bar{x} y - \kappa \bar{x} \bar{y}\right)
\end{aligned}
\tag{3.18}
$$

and

$$
(\tau C'\xi)^\alpha_l \frac{\partial}{\partial \mu^\alpha_l} = \partial_x\,, \qquad \zeta^\beta_j \frac{\partial}{\partial \mu^\beta_j} = \partial_y\,, \qquad (\tau C'\xi)^{\bar{\alpha}}_{\bar{l}} \frac{\partial}{\partial \bar{\mu}^{\bar{\alpha}}_{\bar{l}}} = \partial_{\bar{x}}\,, \qquad \zeta^{\bar{\beta}}_{\bar{j}} \frac{\partial}{\partial \bar{\mu}^{\bar{\beta}}_{\bar{j}}} = \partial_{\bar{y}}\,. \tag{3.19}
$$

Then (3.12) is equivalent to

$$
\partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p \; e^{-A - \frac{1}{2}B}\Big|_{x=\bar{x}=y=\bar{y}=0}\,. \tag{3.20}
$$

This in turn can be easily calculated and gives

$$
\sum_{m=0}^{[p,k]} \mathbf{x}^m \frac{k!p!}{m!} \sum_{l=m}^{[p,k]} (-1)^{l+m} \binom{k}{l} \binom{p}{l} \binom{l}{m} \kappa^{p+k-l-2m} \tag{3.21}
$$

where $[n,m]$ stands for the minimum between n and m. Now we insert this back into the original equation (3.11), we find

$$
\begin{aligned}
|\Lambda_n\rangle * |\Lambda_{n'}\rangle = & \sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{m=0}^{[p,k]}\sum_{l=0}^{[p-m,k-m]} \frac{(-1)^{p+k+l}}{m!} \times \\
& \times \kappa^{n+n'-l-2m} \binom{n}{k}\binom{n'}{p}\binom{k}{m}\binom{k-m}{l}\binom{p}{l+m} \mathbf{x}^m|\mathcal{S}_\perp\rangle\,.
\end{aligned} \tag{3.22}
$$

In order to evaluate these summations we split them as follows

$$
\sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{m=0}^{[p,k]}\sum_{l=0}^{[p-m,k-m]}(\ldots) = \sum_{k=0}^{n}\left(\sum_{p=k+1}^{n'}\sum_{m=0}^{k}\sum_{l=0}^{k-m} + \sum_{p=0}^{k}\sum_{m=0}^{p}\sum_{l=0}^{p-m}\right)(\ldots) \tag{3.23}
$$

Next we replace $l \to l+m$ and (3.23) becomes

$$
\begin{aligned}
\sum_{k=0}^{n}\left(\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=k+1}^{n'} + \sum_{m=0}^{k}\sum_{p=m}^{k}\sum_{l=m}^{p}\right)(\ldots) &= \sum_{k=0}^{n}\left(\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=k+1}^{n'} + \sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{k}\right)(\ldots) = \\
&= \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{n'}(\ldots)
\end{aligned} \tag{3.24}
$$

Summarizing, we have now to calculate

$$
\begin{aligned}
|\Lambda_n\rangle * |\Lambda_{n'}\rangle = & \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{n'} \frac{(-1)^{p+k+l+m}}{m!} \times \\
& \times \kappa^{n+n'-l-m} \binom{n}{k}\binom{n'}{p}\binom{k}{m}\binom{k-m}{l-m}\binom{p}{l} \mathbf{x}^m|\mathcal{S}_\perp\rangle
\end{aligned} \tag{3.25}
$$

Now

$$
\sum_{p=l}^{n'}(-1)^{p+l}\binom{n'}{p}\binom{p}{l} = \binom{n'}{l}\sum_{p=0}^{n'-l}(-1)^p\binom{n'-l}{p} = \binom{n'}{l}(1-1)^{n'-l} \tag{3.26}
$$

This vanishes unless $l = n'$. In the case $n' > n$, $l < n'$. Inserting this into (3.25), for $n' > n$ we get 0.

In the case $n = n'$, l can take the value n'. This corresponds to the case $k = p = l = n = n'$ in eq. (3.25). The result is easily derived

$$
|\Lambda_n\rangle * |\Lambda_n\rangle = \sum_{m=0}^{n}\frac{(-1)^{n+m}}{m!}\binom{n}{m}\kappa^{n-m}\mathbf{x}^m|\mathcal{S}_\perp\rangle = (-\kappa)^n L_n\left(\frac{\mathbf{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle = |\Lambda_n\rangle\,. \tag{3.27}
$$

This proves eq. (1.2).

One could as well derive these results numerically. For instance, in order to obtain (3.27) one could proceed, alternatively, as follows. After setting $n = n'$ in (3.22), one realizes that $|\Lambda_n\rangle * |\Lambda_n\rangle$ has the form

$$
|\Lambda_n\rangle * |\Lambda_n\rangle = \sum_{m=0}^{n} F_m^{(n)}\left(\frac{\mathbf{x}}{\kappa}\right)^m|\mathcal{S}_\perp\rangle \tag{3.28}
$$

where

$$
\begin{aligned}
F_m^{(n)} &= 2\sum_{p=0}^{n-m}\sum_{k=0}^{p}\sum_{l=0}^{k}\frac{(-1)^{p+k+l}\kappa^{2n-l-m}(n!)^2}{(m!)^2(n-k-m)!(n-p-m)!l!(l+m)!(k-l)!(p-l)!} - \\
&\quad -\sum_{p=0}^{n-m}\sum_{l=0}^{p}\frac{(-1)^l\kappa^{2n-l-m}(n!)^2}{[m!(n-p-m)!(p-l)!]^2 l!l+m)!}\,. \qquad (3.29)
\end{aligned}
$$

This corresponds to the desired result if

$$
F_m^{(n)} = \frac{(-1)^{n+m}}{m!}\kappa^n \binom{n}{m}\,. \qquad (3.30)
$$

Using *Mathematica* one can prove (numerically) that this is true for any value of n and m a computer is able to calculate in a reasonable time.

3.2 Proof of eq. (1.3)

The value of the SFT action for any solution $|\Lambda_n\rangle$ is given by

$$
\mathcal{S}(\Lambda_n) = \mathcal{K}\langle\Lambda_n|\Lambda_n\rangle \qquad (3.31)
$$

where $\mathcal{K}$ contains the ghost contribution. As shown in [24], $\mathcal{K}$ is infinite unless it is suitably regularized. Nevertheless, as argued there, $|\Lambda_n\rangle$, together with the corresponding ghost solution, can be taken as a representative of a corresponding class of smooth solutions.

Our task now is to calculate $\langle\Lambda_n|\Lambda_n\rangle$. However it may be important to consider states which are linear combinations of $|\Lambda_n\rangle$. In order to evaluate their action we have to be able to compute $\langle\Lambda_n|\Lambda_{n'}\rangle$. Without loss of generality we can assume $n' > n$. By defining $\tilde{\mathbf{x}} = (a^\dagger\tau C'\xi)\,(a^\dagger\zeta)$ we get

$$
\begin{aligned}
\langle\Lambda_n|\Lambda_{n'}\rangle &= (-\kappa)^{n+n'}\langle\tilde{0}|L_n\left(\frac{\tilde{\mathbf{x}}}{\kappa}\right)e^{-\frac{1}{2}a\tilde{\mathcal{S}}a}L_{n'}\left(\frac{\mathbf{x}}{\kappa}\right)e^{\frac{1}{2}a^\dagger\mathcal{S}a^\dagger}|\tilde{0}\rangle \\
&= L_n\left(\frac{1}{\kappa}(\tau C'\xi)_l^\alpha\zeta_j^\beta\frac{\partial}{\partial\lambda_l^\alpha}\frac{\partial}{\partial\lambda_j^\beta}\right)L_{n'}\left(\frac{1}{\kappa}(\tau\xi)_l^\alpha(C'\zeta)_j^\beta\frac{\partial}{\partial\mu_l^\alpha}\frac{\partial}{\partial\mu_j^\beta}\right)\times \\
&\quad\times\frac{1}{\sqrt{\mathrm{Det}(\mathbb{I}-\mathcal{T}^2)}}\,e^{\lambda C'\frac{1}{\mathbb{I}-\mathcal{T}^2}C'\mu-\frac{1}{2}\lambda C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\lambda-\frac{1}{2}\mu\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}C'\mu}\Big|_{\lambda=\mu=0}\,. \qquad (3.32)
\end{aligned}
$$

For the derivation of this equation, see [11, 14, 16]. Now, let us set

$$
A = \lambda C'\frac{1}{\mathbb{I}-\mathcal{T}^2}C'\mu\,,\qquad B = \lambda C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\lambda\,,\qquad C = \mu\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}C'\mu
$$

and introduce the symbolic notation

$$
(\tau C'\xi)_l^\alpha\frac{\partial}{\partial\lambda_l^\alpha} = \partial_x\,,\qquad \zeta_j^\beta\frac{\partial}{\partial\lambda_j^\beta} = \partial_y\,,\qquad (\tau\xi)_l^\alpha\frac{\partial}{\partial\mu_l^\alpha} = \partial_{\bar{x}}\,,\qquad (C'\zeta)_j^\beta\frac{\partial}{\partial\mu_j^\beta} = \partial_{\bar{y}}\,. \qquad (3.33)
$$

Then, using (3.5) and (3.15), (3.16), we find

$$
\begin{aligned}
&\partial_x\partial_{\bar{x}}A = 0\,, \qquad \partial_x\partial_{\bar{y}}A = -1\,, \qquad \partial_y\partial_{\bar{x}}A = -1\,, \qquad \partial_y\partial_{\bar{y}}A = 0\,. \\
&\partial_x\partial_x B = 0\,, \qquad \partial_x\partial_y B = -2\kappa\,, \qquad \partial_y\partial_y B = 0\,, \\
&\partial_{\bar{x}}\partial_{\bar{x}}C = 0\,, \qquad \partial_{\bar{x}}\partial_{\bar{y}}C = -2\kappa\,, \qquad \partial_{\bar{y}}\partial_{\bar{y}}C = 0\,. \qquad (3.34)
\end{aligned}
$$

We can therefore make the replacement

$$A-\frac{1}{2}B-\frac{1}{2}C\to\kappa xy+\kappa\bar{x}\bar{y}-x\bar{y}-\bar{x}y\,. \tag{3.35}$$

In (3.32) we have to evaluate such terms as

$$\partial_x^k\partial_y^k\partial_{\bar{x}}^p\partial_{\bar{y}}^p(\kappa xy+\kappa\bar{x}\bar{y}-x\bar{y}-\bar{x}y)^{k+p}$$

for any two natural numbers k and p. It is easy to obtain

$$\frac{1}{(p+k)!}\partial_x^k\partial_y^k\partial_{\bar{x}}^p\partial_{\bar{y}}^p(\kappa xy+\kappa\bar{x}\bar{y}-x\bar{y}-\bar{x}y)^{k+p}=\sum_{s=0}^{[p,k]}\binom{k}{s}\binom{p}{s}\,k!\,p!\,\kappa^{p+k-2s}\,. \tag{3.36}$$

Therefore we have

$$\begin{aligned}\langle\Lambda_n|\Lambda_{n'}\rangle&=\sum_{k=0}^{n}\sum_{p=0}^{n'}\frac{(-1)^{k+p}\kappa^{n+n'-p-k}}{k!p!}\binom{n}{k}\binom{n'}{p}\,\partial_x^k\partial_y^k\partial_{\bar{x}}^p\partial_{\bar{y}}^p\,e^{A-\frac{1}{2}B-\frac{1}{2}C}\Big|_{x=y=\bar{x}=\bar{y}=0}\langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle\\&=\sum_{k=0}^{n}\sum_{p=0}^{n'}\frac{(-1)^{k+p}}{k!p!}\binom{n}{k}\binom{n'}{p}\sum_{s=0}^{[p,k]}\binom{k}{s}\binom{p}{s}\,k!p!\kappa^{n+n'-2s}\langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle.\end{aligned} \tag{3.37}$$

As in the previous subsection, we can rearrange the summations as follows,

$$\begin{aligned}\sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{p=0}^{[p,k]}(\dots)&=\sum_{k=0}^{n}\left(\sum_{p=0}^{k}\sum_{s=0}^{p}+\sum_{p=k+1}^{n'}\sum_{s=0}^{k}\right)(\dots)\\&=\sum_{k=0}^{n}\left(\sum_{s=0}^{k}\sum_{p=s}^{k}+\sum_{s=0}^{k}\sum_{p=k+1}^{n'}\right)(\dots)=\sum_{k=0}^{n}\sum_{s=0}^{k}\sum_{p=s}^{n'}(\dots)\,.\end{aligned} \tag{3.38}$$

In conclusion we have to compute

$$\langle\Lambda_n|\Lambda_{n'}\rangle=\sum_{k=0}^{n}\sum_{s=0}^{k}\sum_{p=s}^{n'}\frac{(-1)^{p+k}\,n!n'!}{(n-k)!(n'-p)!(k-s)!(p-s)!(s!)^2}\,\kappa^{n+n'-2s}\langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle\,. \tag{3.39}$$

Now,

$$\sum_{p=s}^{n'}(-1)^p\frac{1}{(n'-p)!(p-s)!}=\sum_{p=0}^{n'-s}\frac{(-1)^{p+s}}{(n'-s)!}\binom{n'-s}{p}=\frac{(-1)^s}{(n'-s)!}(1-1)^{n'-s} \tag{3.40}$$

The right end side vanishes if $n'\neq s$, which is certainly true if $n'>n$. Therefore in such a case, inserting (3.40) into (3.39) we get $\langle\Lambda_n|\Lambda_{n'}\rangle=0$. When $s=n'$, eq. (3.40) is ambiguous. But this corresponds to $p=k=s=n=n'$ in (3.39). The relevant contribution is elementary to compute, and one gets

$$\langle\Lambda_n|\Lambda_n\rangle=\langle\Lambda_0|\Lambda_0\rangle\,. \tag{3.41}$$

This completes the proof of (1.3).

4. The field theory limit and the GMS solitons

In [22] we calculated the low energy limit of $|\Lambda_0\rangle$. This is the Seiberg-Witten limit, [23], defined by $\alpha' \to 0$ with $\alpha' B \gg g$, where g is the closed string metric, in such a way that G, θ and B are kept fixed. It was shown in [22] that the three string vertex (2.11) becomes

$$\mathcal{V}_{00}^{\alpha\beta,rs} \to G^{\alpha\beta}\delta^{rs} - \frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\epsilon^{\alpha\beta}\chi^{rs}\right) \tag{4.1}$$

$$\mathcal{V}_{0n}^{\alpha\beta,rs} \to 0 \tag{4.2}$$

$$\mathcal{V}_{mn}^{\alpha\beta,rs} \to G^{\alpha\beta}V_{mn}^{rs} \tag{4.3}$$

so that the lump state factorizes into two factors, the first involves only the zero modes, while the second contains only non-zero modes. In particular we have

$$\mathcal{S}_{00}^{\alpha\beta} = \frac{2|a|-1}{2|a|+1}G^{\alpha\beta} \equiv s\, G^{\alpha\beta} \tag{4.4}$$

where a has been defined above, (2.14).

Finally, in this limit, the lump state $|\Lambda_0\rangle \equiv |\mathcal{S}\rangle \to |\hat{\mathcal{S}}\rangle$, where

$$|\hat{\mathcal{S}}\rangle = \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{26}\exp\left(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} a_m^{\mu\dagger}S_{mn}a_n^{\nu\dagger}\right)|0\rangle \otimes$$
$$\otimes\frac{4a}{2a+1}\,\frac{b^2}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}}\exp\left(-\frac{1}{2}s a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle\,, \tag{4.5}$$

where $\mu,\nu = 0,\ldots 25$ and $\alpha,\beta = 24,25$ [2]. The norm of the lump is now regularized by the presence of a which is proportional to B, eq. (2.14). Using

$$|x\rangle = \sqrt{\frac{2\sqrt{\mathrm{Det}G}}{b\pi}}\exp\left[-\frac{1}{b}x^\alpha G_{\alpha\beta}x^\beta - \frac{2}{\sqrt{b}}ia_0^{\alpha\dagger}G_{\alpha\beta}x^\beta + \frac{1}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right]|\Omega_{b,\theta}\rangle \tag{4.6}$$

we can calculate the projection onto the basis of position eigenstates of the transverse part of the lump state

$$\langle x|e^{-\frac{s}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle = \sqrt{\frac{2\sqrt{\mathrm{Det}G}}{b\pi}}\frac{1}{1+s}\,e^{-\frac{1-s}{1+s}\frac{1}{b}x^\alpha x^\beta G_{\alpha\beta}}\,. \tag{4.7}$$

Finally, the lump state projected into the x representation is

$$\langle x|\hat{\mathcal{S}}\rangle = \frac{1}{\pi}\exp\Big[-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}\Big]|\Xi\rangle = \frac{1}{\pi}\exp\Big[-\frac{x^\alpha x^\beta\delta_{\alpha\beta}}{\theta}\Big]|\Xi\rangle \tag{4.8}$$

$|\Xi\rangle$ is the sliver state (r.h.s. of first line in eq. (4.5)) and $\theta = \frac{1}{B}$. We recall that B has been chosen nonnegative.

[2] The $a_n^\alpha, a_n^{\alpha\dagger}$ operators must be suitably rescaled in order to absorb the metric factor in the exponent $a_n^{\alpha\dagger}G_{\alpha\beta}S_{nm}a_m^{\beta\dagger}$ of the squeezed state so that it takes the form appropriate for the sliver.

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In order to analyze the same limit for any $|\Lambda_n\rangle$, first of all we have to find the low energy limit of the projectors ρ_1, ρ_2. Also these two projectors factorize into the zero mode and non-zero mode part. The former is given by

$$(\rho_1)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} + i\epsilon^{\alpha\beta}\Big]\,, \qquad (\rho_2)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} - i\epsilon^{\alpha\beta}\Big]\,, \tag{4.9}$$

Now, in order to single out the appropriate limit of $|\Lambda_n\rangle$, we take, in the definition (3.2), $\xi = \hat{\xi} + \bar{\xi}$ and $\zeta = \hat{\zeta} + \bar{\zeta}$, where $\bar{\xi}, \bar{\zeta}$ vanish in the limit $\alpha' \to 0$. Then we make the choice $\hat{\xi}_n = \hat{\zeta}_n = 0,\ \ \forall n > 0$ and determine $\hat{\xi}$ and $\hat{\zeta}$ in such a way that eqs. (3.1), (3.4) and (3.5) are satisfied in the limit $\alpha' \to 0$ (a detailed discussion of this limit is contained in appendix B). In the field theory limit the conditions (3.1) become

$$\hat{\xi}_{0,24} + i\hat{\xi}_{0,25} = 0\,, \qquad \hat{\zeta}_{0,24} + i\hat{\zeta}_{0,25} = 0\,. \tag{4.10}$$

From now on we set $\hat{\xi}_0 = \hat{\xi}_{0,25} = -i\hat{\xi}_{0,24}$ and, similarly, $\hat{\zeta}_0 = \hat{\zeta}_{0,25} = -i\hat{\zeta}_{0,24}$. The conditions (3.5) become

$$\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta \to -\frac{1}{1-s^2}\frac{2}{\sqrt{\mathrm{Det}G}}\hat{\xi}_0\hat{\zeta}_0 = -1 \tag{4.11}$$

$$\xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta \to -\frac{s}{1-s^2}\frac{2}{\sqrt{\mathrm{Det}G}}\hat{\xi}_0\hat{\zeta}_0 = -\kappa\,. \tag{4.12}$$

Compatibility requires

$$\frac{2\hat{\xi}_0\hat{\zeta}_0}{\sqrt{\mathrm{Det}G}} = 1 - s^2\,, \qquad \kappa = s\,. \tag{4.13}$$

At the same time

$$\left(\xi\tau a^\dagger\right)\left(\zeta C' a^\dagger\right) \to -\hat{\xi}_0\hat{\zeta}_0\left(\left(a_0^{24\dagger}\right)^2 + \left(a_0^{25\dagger}\right)^2\right) = -\frac{\hat{\xi}_0\hat{\zeta}_0}{\sqrt{\mathrm{Det}G}}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\,. \tag{4.14}$$

Hermiticity requires that the product $\hat{\xi}_0\hat{\zeta}_0$ be real, in accordance with (4.11), (4.12). Henceforth we will refer to the solutions found in this way as the *factorized solutions*, since, as will become clear in a moment, they realize the factorization of the star product into the Moyal product and Witten's * product. In order to be able to compute $\langle x|\Lambda_n\rangle$ in the field theory limit, we have to evaluate first

$$\begin{aligned}\langle x|\left(a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right)^k e^{-\frac{s}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle &= (-2)^k\frac{d^k}{ds^k}\left(\langle x|e^{-\frac{s}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle\right) \\ &= (-2)^k\frac{d^k}{ds^k}\left(\sqrt{\frac{2\sqrt{\mathrm{Det}G}}{b\pi}}\frac{1}{1+s}e^{-\frac{1-s}{1+s}\frac{1}{b}x^\alpha G_{\alpha\beta}x^\beta}\right).\end{aligned} \tag{4.15}$$

An explicit calculation gives

$$\frac{d^k}{ds^k}\left(\frac{1}{1+s}e^{-\frac{1-s}{1+s}\frac{1}{b}x^\alpha x^\beta G_{\alpha\beta}}\right)\sum_{l=0}^{k}\sum_{j=0}^{k-l}\frac{(-1)^{k+j}}{(1-s)^j(1+s)^{k+1}}\frac{k!}{j!}\begin{pmatrix}k-l-1\\ j-1\end{pmatrix}\langle x,x\rangle^j e^{-\frac{1}{2}\langle x,x\rangle} \tag{4.16}$$

where we have set

$$\langle x, x\rangle = \frac{1}{ab} x^\alpha G_{\alpha\beta} x^\beta = \frac{2r^2}{\theta} \tag{4.17}$$

with $r^2 = x^\alpha x^\beta \delta_{\alpha\beta}$. In this equation it must be understood that, by definition, the binomial coefficient $\binom{-1}{-1}$ equals 1.

Now, inserting (4.16) in the definition of $|\Lambda_n\rangle$, we obtain after suitably reshuffling the indices:

$$\begin{aligned}\langle x|(-\kappa)^n L_n\Big(\frac{\mathbf{x}}{\kappa}\Big) e^{-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle \to\ & \langle x|(-s)^n L_n\Big(-\frac{1-s^2}{2s}\, a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\Big) \times \\ & \times e^{-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle = \\ = & \frac{(-s)^n}{(1+s)} \sum_{j=0}^{n}\sum_{k=j}^{n}\sum_{l=j}^{k} \binom{n}{k}\binom{l-1}{j-1} \frac{1}{j!}\frac{(1-s)^k}{(1+s)^j s^k} \times \\ & \times (-1)^j \langle x, x\rangle^j\, e^{-\frac{1}{2}\langle x,x\rangle} \sqrt{\frac{2\sqrt{\mathrm{Det}G}}{b\pi}}\,. \end{aligned} \tag{4.18}$$

The expression can be evaluated as follows. First one uses the result

$$\sum_{l=j}^{k} \binom{l-1}{j-1} = \binom{k}{j}. \tag{4.19}$$

Inserting this into (4.18) one is left with the following summation, which contains an evident binomial expansion,

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j}\Big(\frac{1-s}{s}\Big)^k = \binom{n}{j}\frac{(1-s)^j}{s^n}\,. \tag{4.20}$$

Replacing this result into (4.18) we obtain

$$\langle x|(-\kappa)^n L_n\Big(\frac{\mathbf{x}}{\kappa}\Big) e^{-\frac{1}{2}s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle \to \frac{2|a|+1}{4|a|}\sqrt{\frac{2\sqrt{\mathrm{Det}G}}{b\pi}}(-1)^n \sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\Big(-\frac{2r^2}{\theta}\Big)^j e^{-\frac{r^2}{\theta}}.$$

Recalling now that the definition of $|\hat{\mathrm{S}}\rangle$ includes an additional numerical factor (see eq. (4.5)), we finally obtain

$$\begin{aligned}\langle x|\Lambda_n\rangle \to \langle x|\hat{\Lambda}_n\rangle &= \frac{1}{\pi}(-1)^n \sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\Big(-\frac{2r^2}{\theta}\Big)^j e^{-\frac{r^2}{\theta}}|\Xi\rangle \\ &= \frac{1}{\pi}(-1)^n L_n\Big(\frac{2r^2}{\theta}\Big) e^{-\frac{r^2}{\theta}}|\Xi\rangle \end{aligned} \tag{4.21}$$

as announced in the introduction. The coefficient in front of the sliver $|\Xi\rangle$ is the $n-th$ GMS solution. Strictly speaking there is a discrepancy between these coefficients and the corresponding GMS soliton, given by the normalizations which differ by a factor of 2π.

This can be traced back to the traditional normalizations used for the eigenstates $|x\rangle$ and $|p\rangle$ in the SFT theory context and in the Moyal context, respectively. This discrepancy can be easily dealt with with a simple redefinition.

Finally we remark that, for the solutions $\langle x|\hat{\Lambda}_n\rangle$, in view of the properties of the GMS solitons, we have achieved factorization of the star product with B field into the Moyal product and Witten's star product.

5. Conclusion

In [12] it was shown that a generic noncommutative scalar field theory with polynomial interaction allows for solitonic solutions in any space dimension, see also [13] and references therein. The solutions are very elegantly constructed in terms of harmonic oscillators eigenstates $|n\rangle$. In particular, solitonic solutions correspond to projectors $P_n = |n\rangle\langle n|$. Via the Weyl transform these projectors can be mapped to classical functions $\psi_n(x, y)$ of two variables x, y, in such a way that the operator product in the Hilbert space correspond to the Moyal product in (x, y) space.

This construction is rather universal and does not depend in any essential way on the form of the potential. Now, as we have noticed in the introduction, the low energy effective tachyonic field theory derived from SFT in the presence of a background B field is a noncommutative scalar field theory of the type described above. Therefore it is endowed with the GMS noncommutative solitons. It is reasonable to expect that these solitons may emerge from soliton-type solutions of the SFT, which has the noncommutative scalar tachyonic field theory as its low energy effective action. Therefore the low energy GMS solitons we found in the previous sections are no surprise. What is rewarding however is the isomorphism we find between the lump solutions $|\Lambda_n\rangle$ in VSFT and the corresponding GMS solitons. Setting $r^2 = x^2 + y^2$ and $\psi_n(x, y) = 2(-1)^n\, L_n(\frac{2r^2}{\theta})\, e^{-\frac{r^2}{\theta}}$, we have in fact the following correspondences

$$\begin{array}{ccccc} |\Lambda_n\rangle & \longleftrightarrow & P_n & \longleftrightarrow & \psi_n(x,y) \\ |\Lambda_n\rangle * |\Lambda_{n'}\rangle & \longleftrightarrow & P_n P_{n'} & \longleftrightarrow & \psi_n \star \psi_{n'} \end{array} \tag{5.1}$$

where $\star$ denotes the Moyal product. Moreover

$$\langle\Lambda_n|\Lambda_{n'}\rangle \longleftrightarrow \mathrm{Tr}(P_n P_{n'}) \longleftrightarrow \int dx dy\, \psi_n(x,y)\psi_{n'}(x,y) \tag{5.2}$$

up to normalization (see (1.3)). This correspondence seems to indicate that the Laguerre polynomials hide a universal structure of these noncommutative algebras.

It is evident from the above that the GMS solitons are the low energy remnants of corresponding D-branes in SFT. This explains many features of the former: why, for instance, the energy of the soliton given by $\sum_{k=0}^{n-1} |k\rangle\langle k|$ is n time the energy of the soliton $|0\rangle\langle 0|$; this is nothing but a low energy relic of the same property for the tensions of the corresponding D-branes.

A. Consistency conditions

In this appendix we discuss compatibility among eqs. (3.1), (3.4) and (3.5). In particular we would like to argue that not only they admit solutions ξ and ζ but that the latter are expected to have an infinite number of undetermined components. To start with, from [16] we deduce that the ρ projectors, in the absence of a B field, halve the number of ξ and ζ components. This will remain true in the presence of the deformation due to the B field (at least for generic values of B, but there is no evidence of 'critical' values of B (see [22] for exact calculations in this sense)). Second, the equation (3.4) relates ξ^*, ζ^* to ξ, ζ. Third, the eqs. (3.5) is a finite set of conditions. Therefore, in a generic situation, we expect that there exist ξ and ζ solutions to these equations with infinite many indeterminate components. The only trouble could come from a mutual incompatibility of these equations. Let us discuss this point more closely by first solving explicitly eq. (3.4). This equation has two simple solutions: (I) $\zeta = \tau\xi^*$ and (II) $\xi = C'\xi^*$, $\zeta = C'\zeta^*$. Let us analyze their compatibility with the remaining equations. To do so we have to first recall eqs. (2.15), (2.16), (2.17) and (2.22), (2.23), (2.24), (2.25). From the properties quoted in section 3 of [21] one easily gets in addition

$$(\mathcal{V}^{rs})^* = \tilde{\mathcal{V}}^{rs} \,. \tag{A.1}$$

For clarity we recall that the * superscript denotes complex conjugation, $\tilde{}$ represents transposition with respect to the α, β indices, while ^{T} is the transposition with respect to both α, β and N, M indices. We also introduce the operation $^\dagger = {}^{*T}$. Using all this it is easy to prove that

$$\begin{aligned} (\mathcal{X}^{rs})^* &= \tilde{\mathcal{X}}^{rs}\,, \quad \text{i.e.} \quad (\mathcal{X}^{rs})^\dagger = \mathcal{X}^{rs} \\ \left[(\mathcal{X}^{rs})^*, \left(\mathcal{X}^{r's'}\right)^*\right] &= 0\,, \qquad \left[\tilde{\mathcal{X}}^{rs}, \tilde{\mathcal{X}}^{r's'}\right] = 0\,, \\ \rho_i^\dagger &= \rho_i\,, \quad \text{i.e.} \quad \rho_i^* = \tilde{\rho}_i\,, \quad i = 1,2 \\ \tau\rho_i &= \tilde{\rho}_i\tau\,, \qquad i = 1,2\,. \end{aligned} \tag{A.2}$$

Now we are ready to verify compatibility of (I) and (II) with eqs. (3.1), (3.5). As for the former we have

$$\rho_1\zeta = \rho_1\tau\xi^* = \tau\tilde{\rho}_1\xi^* = \tau\rho_1^*\xi^* = \tau(\rho_1\xi)^* = 0\,. \tag{A.3}$$

Therefore (I) is consistent with (3.1). In case (II) we have instead

$$\rho_1\xi = \rho_1 C'\xi^* = C'\tilde{\rho}_2\xi^* = C'\rho_2^*\xi^* = C'\xi^* \tag{A.4}$$

so that requiring $\rho_1\xi = 0$ implies $\xi = 0$ identically. Therefore we must discard solution (II).

Next we have to prove compatibility of (I) with (3.5), in other words we must show that when replacing (I) into the r.h.s. of the two eqs. (3.5) we get real numbers. Let us show this for the first equation, because for the second no significant modification is needed

$$\left(\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta\right)^* = \xi^{T*}\tau\frac{1}{\mathbb{I}-(\mathcal{T}^*)^2}\zeta^* = \zeta^T\frac{1}{\mathbb{I}-(\mathcal{T}^*)^2}\tau\xi = \xi^T\tau\frac{1}{\mathbb{I}-(\mathcal{T}^\dagger)^2}\zeta = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta$$

where the second equality is obtained by replacement of (I), and the third by transposition.

To complete our argument we have to prove compatibility of eq. (3.1) with (3.5). Let us impose that ξ and ζ are solutions to (3.1),

$$\xi^T \tau \frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = \xi^T \tau \frac{1}{\mathbb{I}-\mathcal{T}^2}\rho_2\zeta = \xi^T \tau\rho_2 \frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = \xi^T \tilde{\rho}_2\tau \frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = \xi^T \tau \frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta \tag{A.5}$$

where the second equality is a consequence of $[\rho_i, \mathcal{T}] = 0$, while the last follows from

$$\xi^T = (\rho_2\xi)^T = \xi^T \rho_2^T = \xi^T \tilde{\rho}\,. \tag{A.6}$$

We see that no additional constraints are obtained.

It remains for us to examine the question of whether the solution (3.6) to (3.4) is the only possible one. We do not have a formal proof of this fact, but we can produce the following argument. On the l.h.s. of eq. (3.4) the vectors $\tau\xi^*$ and $C'\zeta^*$ lie in the subspaces annihilated by ρ_1 and ρ_2, respectively. Similarly on the r.h.s. of the same equation $\tau C'\xi$ and ζ lie in subspaces annihilated by ρ_2 and ρ_1, respectively. Since the subspaces annihilated by ρ_1 and ρ_2 are linearly independent, the equality of the two sides of eq. (3.4) should automatically imply eq. (3.6) (up to an overall constant factor which can be absorbed by rescaling ζ and ξ in opposite directions) [3].

B. The $\alpha' \to 0$ limit

This appendix is devoted to a discussion of the factorized $\alpha' \to 0$ limit found after eq. (4.9). There we wrote the vectors ξ and ζ in the form $\xi = \hat{\xi} + \bar{\xi}$ and $\zeta = \hat{\zeta} + \bar{\zeta}$, where $\bar{\xi}, \bar{\zeta}$ are supposed to vanish in the limit $\alpha' \to 0$. Then we made the choice $\hat{\xi}_n = \hat{\zeta}_n = 0, \ \forall n > 0$ and determined $\hat{\xi}$ and $\hat{\zeta}$ in such a way that eqs. (3.1), (3.4), (3.5) be satisfied in the limit $\alpha' \to 0$. We called such kind of solutions the factorized ones. In section 4 it was tacitly assumed that such solutions are $\alpha' \to 0$ limits of $\alpha' \neq 0$ solutions. This is a reasonable assumption, based on the fact that the $\alpha' \to 0$ limit is smooth in all the relevant equations (we mean specifically eqs. (3.1), (3.5)). The latter are a system of algebraic (linear and quadratic) equations, and we do not expect any kind of singularity in the $\alpha' \to 0$ limit, and, therefore, the factorized solutions should indeed be the limits of solutions to eqs. (3.1), (3.4), (3.5) for generic α'.

Here we would like to examine this question more closely. To confirm our conjecture based on smoothness we will explicitly construct $\bar{\xi}$ and $\bar{\zeta}$ as power series which satisfy eqs. (3.1), (3.4), (3.5) and vanish for $\alpha' \to 0$. We recall from [18] that the most appropriate way to define the $\alpha' \to 0$ limit is to introduce a small dimensionless parameter ϵ together with the rescalings

$$\begin{aligned} V_{00} &\to \epsilon^2 V_{00} \\ V_{0n} &\to \epsilon V_{0n} \\ V_{mn} &\to V_{mn}\,. \end{aligned} \tag{B.1}$$

[3] We thank one of the referees of this paper for suggesting this argument.

From now on we set α' to a constant value, say 1, and consider instead small ϵ expansions. Let f be a generic function of α', like $\rho_i, \xi, \mathcal{X}, \ldots$, and write it as

$$f = \hat{f} + \bar{f} = \hat{f} + \epsilon f' + \epsilon^2 f'' + \ldots + \epsilon^n f^{(n)} + \ldots . \tag{B.2}$$

When convenient we will use the notation $f^{(0)}$ for $\hat{f}$. We wish to satisfy eqs. (3.1), (3.4), (3.5) in this ϵ-expanded form. As for (3.4), we understand that it is satisfied via (3.6). Next we deal with (3.1). Writing $\rho_1^2 = \rho_1$ in the expanded form, we find

$$\begin{aligned} \hat{\rho}_1^2 &= \hat{\rho}_1 \\ \rho_1' &= \hat{\rho}_1 \rho_1' + \rho_1' \hat{\rho}_1 \\ \rho_1'' &= {\rho_1'}^2 + \hat{\rho}_1 \rho_1'' + \rho_1'' \hat{\rho}_1 \,, \end{aligned} \tag{B.3}$$

$$\ldots$$

$$\rho_1^{(n)} = \sum_{k=0}^{n} \rho_1^{(k)} \, \rho_1^{(n-k)} . \tag{B.4}$$

Similarly, from $\rho_1 \xi = 0$, we get

$$\hat{\rho}_1 \hat{\xi} = 0 \tag{B.5}$$

$$\hat{\rho}_1 \xi' + \rho_1' \hat{\xi} = 0 \tag{B.6}$$

$$\rho_1'' \hat{\xi} + \rho_1' \xi' + \hat{\rho}_1 \xi'' = 0 \,, \tag{B.7}$$

$$\ldots$$

$$\sum_{k=0}^{n} \rho_1^{(k)} \xi^{(n-k)} = 0 . \tag{B.8}$$

A solution to (B.5) was found in section 4 (the factorized solution). If we replace the second of eqs. (B.3) into (B.6) we get $\hat{\rho}_1(\xi' + \rho_1' \hat{\xi}) = 0$ which implies that

$$\xi' = -\rho_1' \hat{\xi} + \eta' \tag{B.9}$$

where η' is any solution to the equation $\hat{\rho}_1 \eta' = 0$; recalling appendix A, η' will have an infinite number of undetermined components. Analogously, for ξ'' we find

$$\xi'' = -\rho_1'' \hat{\xi} - \rho_1' \xi' + \eta'' \tag{B.10}$$

where η'' is a new arbitrary solution to $\hat{\rho}_1 \eta'' = 0$. We remark that a term proportional to ${\rho_1'}^2 \hat{\xi}$ could be added to the r.h.s. of eq. (B.10). This follows from the fact that such a term is in the kernel of $\hat{\rho}_1$, for

$$\hat{\rho}_1 \rho_1' \rho_1' \hat{\xi} = \left(\rho_1' \rho_1' - \rho_1' \hat{\rho}_1 \rho_1'\right) \hat{\xi} = \left(\rho_1' \left(\rho_1' \hat{\rho}_1 + \hat{\rho}_1 \rho_1'\right) - \rho_1' \hat{\rho}_1 \rho_1'\right) \hat{\xi} = 0 \tag{B.11}$$

where we have used the relation $\rho_1' = \rho_1' \hat{\rho}_1 + \hat{\rho}_1 \rho_1'$ twice; the first time applied to the product $\hat{\rho}_1 \rho_1'$ in the l.h.s. , the second time in an obvious way. Therefore the addition of such a term simply amounts to a redefinition of η''.

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Proceeding further as above, it is evident that at every new order of approximation for ξ we get in addition a new arbitrary eigenvector of $\hat{\rho}_1$ with eigenvalue 0. In general

$$\xi^{(n)} = -\sum_{k=0}^{n-1} \rho_1^{(n-k)} \xi^{(k)} + \eta^{(n)} \tag{B.12}$$

where $\eta^{(n)}$ is the eigenvector in question. It is possible to construct explicit examples of ξ at the lowest orders in ϵ: for instance $\xi = \rho_2 \hat{\xi}$ satisfies the above requirements to order zero and 1 in ϵ.

Now let us deal with (3.5). We first call

$$F = \frac{1}{\mathbb{I} - \mathcal{T}^2} \qquad G = \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2}$$

then, as above, expand the l.h.s. of (3.5) in powers of ϵ. We recall from section 4 that by construction

$$\hat{\xi}\tau \hat{F} \hat{\zeta} = -1 \,, \qquad \hat{\xi}\tau \hat{G} \hat{\zeta} = -\kappa \,.$$

Therefore the higher orders in the l.h.s. of (3.5) must be equated to 0. To first order in ϵ, after explicitly inserting (3.6), we get

$$\hat{\xi}\tau F' \tau \hat{\xi}^* - \hat{\xi}\tau \hat{F} \tau \rho_1'^{\,*} \hat{\xi}^* - \hat{\xi} \rho_1'^{\,T} \tau \hat{F} \tau \hat{\xi}^* + \hat{\xi}\tau \hat{F} \tau \eta'^* + \eta' \tau \hat{F} \tau \hat{\xi}^* = 0 \tag{B.13}$$

and an analogous equation with F everywhere replaced by G. In order to understand how these two equations can be satisfied we have to go back to (B.1). Inserting the latter in (2.11), one can see that the first order correction to $\mathcal{V}_{00}^{rs}$ vanishes. The same holds for $\mathcal{X}_{00}^{rs}$, and, due to the block diagonal form of the hatted objects $\hat{\mathcal{X}}^{rs}, \hat{\rho}_i, \hat{F}, \ldots$, we come to the conclusion that $(\rho_1')_{00} = (F')_{00} = (G')_{00} = 0$ (in fact the first nonvanishing corrections to the $M = 0, N = 0$ components come from the second order in ϵ). Now recalling that the only nonvanishing component of $\hat{\xi}$ is the zeroth one, we see that in (B.13) the first three terms vanish. Therefore (B.13) is satisfied provided we choose $\eta_0' = 0$. The same is true for the analogous equation with F replaced by G.

This proves that, to first order in ϵ, eqs. (3.1), (3.4), (3.5) can be satisfied with $\xi' = \eta'$, where η' has vanishing zeroth components, while η_n' are subject to the condition of defining an eigenvectors of $\hat{\rho}_1$ with 0 eigenvalue, but are otherwise arbitrary. We remark that the precise form (B.1) of the rescaling plays a crucial role in satisfying (B.13) and its companion equation: were the first three terms of (B.13) nonvanishing, it would be impossible to satisfy both equations for any value of s because $\hat{G}_{00} = s\hat{F}_{00}$.

Let us analyze now the second order approximation in ϵ. The equation analogous to (B.13) is

$$\xi'' \tau \hat{F} \hat{\zeta} + \hat{\xi} \tau F'' \hat{\zeta} + \hat{\xi} \tau \hat{F} \zeta'' + \xi' \tau F' \hat{\zeta} + \xi' \tau \hat{F} \zeta' + \hat{\xi} \tau F' \zeta' = 0 \tag{B.14}$$

From the above discussion we know that in the first three terms only the zeroth components of ξ'' and ζ'' and the $_{00}$ component of F'' do contribute. However here no such simplification occurs as in (B.13), because all the components of $\xi', \zeta' = \tau\xi'^*$ and F' will contribute (and the equality $\hat{G}_{00} = s\hat{F}_{00}$ is not dangerous in this new context). But now we have at our disposal all the infinite many arbitrary components of η' to satisfy (B.14) and the companion equation with F replaced by G. On the basis of the argument introduced in appendix A, this is possible in an infinite number of ways.

Now the pattern is clear. At the third order in ϵ we can count on the infinite many arbitrary components of η'' to satisfy the third order approximation of (3.5), and so on. This completes our argument to show that the factorized solutions can be obtained as the $\alpha' \to 0$ limit of $\alpha' \neq 0$ solutions of eqs. (3.1), (3.4), (3.5).

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